

THE SPACE-TIME TORSION IN THE CONTEXT OF THE EXACT  
FOLDY-WOUTHUYSEN TRANSFORMATION FOR A DIRAC FERMION

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**Abstract**

In this work we focus our attention in the inconsistency that appears when the Semi-Exact Foldy Wouthuysen transformation for the Dirac field interacting with space-time torsion field is performed. In order to solve this problem, we present a new involution operator that makes possible to perform the exact transformation when torsion field is present. Such operator has a structure, well known in the literature, composed of the product of an operator that acts in the matrices space and another one that acts in the function space. We also present the bound state of this theory and discuss the possible experimental analysis.

**Keywords:** Dirac equation, CPT-Lorentz violating terms, Exact Foldy-Wouthuysen transformation.

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## 1 Introduction

It is known that torsion fields arises when one takes into account the Gauge approach to gravity [1, 2] and this subject has received considerable attention of the scientific community. In [3], for example, it is possible to find a review about the renormalization properties of quantum field theories in curved space-time in the presence of CPT-Lorentz violating terms. Is also possible to find recent works (see [4, 5] and references cited therein) that treat the possibility of CPT-Lorentz symmetry breaking in a more phenomenological point of view. Although there is not yet concise experimental evidences of torsion fields, it has been the aim of several recent studies (see [6, 7, 8, 9] and references cited therein). Spin-torsion discussion in the context of classical and quantum effects are well described in [10, 11].

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As pointed out in [6], high energy level is more essential in the sense of getting torsion experimental manifestations. A concise review about Foldy-Wouthyusen Transformation (FWT) [12] and semi-classical limit for relativistic particles in strong external fields can be found in [13]. Reference [12] also shows that FWT has succeeded in providing detailed information about the nonrelativistic approximation. However, there is a considerable advantage in performing the Exact Foldy-Wouthyusen Transformation (EFTW) [6, 14, 15, 16, 17, 18]. The reason is that, even EFTW is more complex [17, 18], it presents some additional terms that can be missed if one uses FWT.

The magnitude of the coupling constant of the torsion field with the Dirac spinor is very small [11] and some features are specially related to a concise study of the nonrelativistic limit of the Dirac equation in the presence of an external torsion field. The case of the Dirac field interacting with many possible external fields associated with CPT-Lorentz violation was developed in the recent paper [19], where the authors perform the EFTW together a review of the connection between CPT-Lorentz violating terms and phenomenology.

Although the nonrelativistic limit was studied for the Dirac field interacting with the set of possible external fields (the torsion field could not be included in that set of external fields) associated with CPT-Lorentz violation [19], there is not in the literature a concise study of the interaction of the Dirac field and the vectorial part of the torsion, in the context of the EFTW. In order to understand the reason for this we should mention that the possibility of performing the exact transformation depends on the fact that the commutation relation between the so called involution operator and the torsion field should be satisfied. One can check that for the torsion field, it is not. The first attempt in order to perform EFTW is known as Semi Exact Foldy-Wouthyusen Transformation (SEFTW) [20]. Such approach imposes some changes in the action of the theory, but it seems to work very well for several cases [21] and its results are in accordance with usual EFTW. However, when the torsion field is considered, SEFTW seems to fail<sup>5</sup>.

In the present paper we consider the combined action of torsion and strong magnetic field on the massive spinor field and on the corresponding particle. In this case, the Hamiltonian does not admit the EFTW in the usual way. We discuss a method that enables one to perform EFTW and get physical results of this situation. We begin with the correct choice of involution operator. Such choice is quite natural in the sense that is in accordance with [18, 22]. The basic idea is to work with an operator that acts only on the part of term that is not anti-commuting with the usual evolution operator. Taking into account that torsion breaks parity, a reasonable possibility would be to impose a new involution operator that has two contributions, the first one acts on matrices and another one on the functions. The second contribution of this operator (the new one) compensates the fact that the matrices commute with the involution operator.

The method is used with the torsion field, but it can be straightforwardly generalized to other terms. We emphasize that the method itself is the main results here, in the sense that it opens the window to the possibility of performing the EFTW for some cases that are until now, not contemplated by the literature and extract from its physical information. Experimental perspectives are also analyzed.

The paper is organized as follows. In section 2 we present a brief review about SEFTW and

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<sup>5</sup>SEFTW consideration presents some not understandable physical impositions, for torsion case.

discuss some restrictions of this model when torsion field is considered. In section 3, the new proposal to perform EFWT for the Dirac field interacting with space-time torsion is presented. Sections 4 and 5 are devoted to the equations of motion and the study of the bound state of the theory, respectively. In section 6 we draw our conclusions. Throughout the paper we use Greek letters for the indexes which run from 0 to 3. Latin indexes are used for the space coordinates and run from 1 to 3.

## 2 Semi Exact Foldy-Wouthuysen Transformation

We present in this section a brief review about SEFWT [20, 21]. Consider the spin-1/2 particle in an external torsion and electromagnetic fields. We are going to consider the magnetic and torsion fields which can only vary with time, but do not depend on the space coordinates. The Hamiltonian we shall deal with is written as follows

$$\mathcal{H} = c\vec{\alpha} \cdot \vec{p} - e\vec{\alpha} \cdot \vec{A} - \eta_1 \vec{\alpha} \cdot \vec{S} \gamma_5 + e\Phi + \eta_1 \gamma_5 S_0 + mc^2 \beta. \quad (1)$$

Here we used notations  $A_\mu = (\Phi, \vec{A})$ ,  $S_\mu = (S_0, \vec{S})$ . In case of constant magnetic field, one can set  $\Phi = 0$ . We adopt notations as described in [23] for Dirac Matrices and also denote the  $\gamma^0$  Dirac matrix as  $\beta$ .

Only those theories where the Hamiltonian obey the following relation, enable one to perform the EFWT [14, 17, 18, 24].

$$J\mathcal{H} + \mathcal{H}J = 0, \quad (2)$$

where  $J = i\gamma_5\beta$ . The quantity  $J$  is the so called involution operator, which is Hermitian and unitary. It is also known that  $J\beta + \beta J = 0$ .

Direct inspection show that the term  $\eta_1 \vec{\alpha} \cdot \vec{S} \gamma_5$  is the only one in the Hamiltonian (1) that does not satisfy the condition (2). From this point of view, a natural conclusion is that would not be possible to perform EFWT when one take into account the torsion field in the Hamiltonian of the theory. However, there is a possible consideration that modifies this scenario, in some sense (see eg. [20] and references cited therein). Let us make an *ad hoc* modification. According to this modification, the term commented above should be multiplied by the  $\beta$ -matrix. Observe that such modification satisfies the condition (2) and now the EFWT is perfectly possible<sup>6</sup>. After all, the Hamiltonian we are going to deal with has the form

$$\mathcal{H} = c\vec{\alpha} \cdot \vec{p} - e\vec{\alpha} \cdot \vec{A} - \eta_1 \vec{\alpha} \cdot \vec{S} \gamma_5 \beta + \eta_1 \gamma_5 S_0 + mc^2 \beta. \quad (3)$$

According to the standard EFWT prescription [17, 19], the next step is to obtain  $H^2$ . Direct calculations give the result

$$\begin{aligned} \mathcal{H}^2 &= (c\vec{p} - e\vec{A} - \eta_1 \vec{\Sigma} S_0)^2 + m^2 c^4 + 2\eta_1 mc^2 \vec{\Sigma} \cdot \vec{S} \\ &- (\eta_1)^2 (\vec{S})^2 - \hbar ce \vec{\Sigma} \cdot \vec{B} - 2(\eta_1)^2 (S_0)^2 + i\eta_1 \gamma_5 \beta \vec{\Sigma} \cdot [\vec{S} \times (c\vec{p} - e\vec{A})]. \end{aligned} \quad (4)$$

Observe that the last term in this equation transforms (under parity) in a different way compared to the other terms in the Hamiltonian. However, there is no reasonable physical arguments that

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<sup>6</sup>In the linear order in the torsion field, an extra  $\beta$  has no effect.

enable us to suppose that  $\vec{\Sigma} \cdot [\vec{S} \times (c\vec{p} - e\vec{A})] = 0$ . From this point, the next step is to perform the exact transformation, We shall not to describe this procedure in details here (standard procedure is described in [20, 24]. The transformed Hamiltonian is written as follows

$$\begin{aligned}\mathcal{H}^{tr} = & \beta mc^2 + \frac{\beta}{2mc^2}(c\vec{p} - e\vec{A} - \eta_1 \vec{\Sigma} S_0)^2 + \beta \eta_1 \vec{\Sigma} \cdot \vec{S} \\ & - \beta \frac{\hbar e}{2mc} \vec{\Sigma} \cdot \vec{B} - \beta \frac{(\eta_1)^2}{mc^2} (S_0)^2 + i\beta \eta_1 \gamma_5 \beta \vec{\Sigma} \cdot [\vec{S} \times (c\vec{p} - e\vec{A})].\end{aligned}\quad (5)$$

Now and so on we denote the terms with "tr" index as the transformed ones and such terms belong to the final transformed Hamiltonian. Taking into account the two components spinor

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{\frac{-ime^2 t}{\hbar}} \quad (6)$$

and writing the Dirac equation in the Schrödinger form  $i\hbar\partial_t\psi = \mathcal{H}\psi$ , the Hamiltonian for  $\varphi$  is written in the following way

$$\mathcal{H}_\varphi^{tr} = \frac{1}{2m}(\vec{\Pi})^2 + B_0 + \vec{\sigma} \cdot \vec{Q}, \quad (7)$$

where

$$\begin{aligned}\vec{\Pi} &= \vec{p} - \frac{e}{c}\vec{A} - \frac{\eta_1}{c}S_0\vec{\sigma} \quad , \quad B_0 = -\frac{(\eta_1)^2}{mc^2}(S_0)^2 \\ \vec{Q} &= \eta_1\vec{S} - \frac{\hbar e}{2mc}\vec{B} + \frac{\eta_1}{mc}\vec{S} \times (\vec{p} - \frac{e}{c}\vec{A}).\end{aligned}$$

The canonical quantization of (7) lead us to (quasi) classical equations of motion

$$\frac{dx_i}{dt} = \frac{1}{m}\left(p_i - \frac{e}{c}A_i - \frac{\eta_1}{c}\sigma_i S_0\right) + \frac{\eta_1}{mc}[\vec{\sigma} \times \vec{S}]_i = v_i \quad (8)$$

$$\frac{dp_i}{dt} = \frac{1}{m}\left(p^j - \frac{e}{c}A^j - \frac{\eta_1}{c}\sigma^j S_0\right)\frac{e}{c}\frac{\partial A_j}{\partial x^i} + \frac{\eta_1}{mc}[\vec{\sigma} \times \vec{S}]^j \frac{e}{c}\frac{\partial A_j}{\partial x^i} \quad (9)$$

$$\frac{d\sigma_i}{dt} = [\vec{R} \times \vec{\sigma}]_i \quad , \quad R_j = \frac{2\eta_1}{\hbar}\left[S_j - \frac{1}{c}v_j S_0 + \left(S \times \frac{\vec{v}}{c}\right)_j + \frac{2\eta_1}{\hbar}S_0(\vec{S} \times \vec{\sigma})_j\right] + \frac{e}{mc}B_j. \quad (10)$$

Combining these last equation, the Lorentz force is written as

$$m\frac{dv_i}{dt} = -\frac{e}{c}\frac{\partial A_i}{\partial t} + \frac{e}{c}[\vec{v} \times \vec{B}]_i - \frac{\eta_1}{c}\sigma_i\frac{\partial S_0}{\partial t} - \frac{\eta_1}{c}\frac{\partial(\vec{S} \times \vec{\sigma})_i}{\partial t} \quad (11)$$

Based on what was explained above, one can tend to suppose that the SEFWT approach is not consistent. On the other hand, the SEFWT is performed in [21] for several cases and the results are in accordance with usual EFWT. Although the approach itself seems not to have inconsistencies, it fails, in the practical sense for the case studied here. In order to get a better perspective about this situation, we present in the next section a new proposal to perform the EFWT transformation for the space time torsion case.

### 3 Exact Foldy-Wouthuysen Transformation, the new proposal

We present here an approach that enables one to work with the usual EFWT for the torsion field. The main idea is to consider a more general involution operator form rather than the one used in the previous section. We shall consider the more general involution operator structure [18, 22]

$$J = M \times \hat{F}, \quad (12)$$

where  $M$  and  $\hat{F}$  are operators that act on the matrices and functions (external fields in the action for example) space respectively. With this assumption the general form of the Hamiltonian (1) is not changed. The involution operation we shall deal with has the following explicit form

$$J = i\gamma^5 \beta \hat{P} \quad (13)$$

where  $\hat{T}$  is time reverse operator and  $\hat{P}$  the parity operator.

One can find in the introduction of [25] a list of references to CPT theorem. It is important to remember some basic relations for the parity reflection  $\hat{P}$  and time reversal  $\hat{T}$  that are important for us in this work for quadri-vectors. The important thing here is to take into account how the vectors and pseudo-vectors respond to the action of these operators. The main point is that under T-transformation only the time component of the four vector changes sign and for the P-transformation the vector part is affected. For a pseudo-vector, like  $S_\mu$ , the situation is that if  $x'_i \rightarrow -x$  (parity), the vector part changes sign and if  $t' \rightarrow -t$ , the  $S_0$  part changes sign. As it should be, since we don't have C-symmetry breaking terms of this Hamiltonian, for this case the transformation PT will give the covariance of the Hamiltonian.

Therefore, what we are proposing here that can be considered a new approach is a method to find the correct form of the involution operator that allows the EFWT method to be applied in some cases it would not be possible. Here, the involution operator (13), that has the same form, for example, in ([18]), does not restrict the form of the external analyzed field, as it was done for the electromagnetic potential vector on the cited work. The idea here is applied only for possible CPT/Lorentz symmetry breaking terms. One should know which kind of symmetry the studied term breaks, before the calculations (from the literature). In our case we have parity, for torsion, as an example. Then the next step is to propose a form for the operator  $\hat{F}$  in (12) that is  $\hat{P}$ , in our case.

Observe now that the commutation relation (2) is obeyed, when one take into account relations (13) and the Hamiltonian of the system (1). For this reason, EFWT is completely possible to be performed. It is worth noting that in a general case, if one wants to perform the exact transformation or any external term, what need to be done is to find the explicit form for operator  $\hat{F}$ . It must be done to show the consistence of the method.

At this point we are able to perform the EFWT. The procedure we use is the standard one, which is well described in [19, 20, 24]. The transformed Hamiltonian, for the Dirac spinor is written as follows

$$\begin{aligned} \mathcal{H}^{tr} &= \beta mc^2 + \frac{\beta}{2mc^2} (c\vec{p} - e\vec{A} - \eta_1 \vec{\Sigma} S_0 - \eta_1 \gamma_5 \vec{S})^2 + \beta \eta_1 \vec{\Sigma} \cdot \vec{S} \\ &- \beta \frac{\hbar e}{2mc} \vec{\Sigma} \cdot \vec{B} - \beta \frac{(\eta_1)^2}{mc^2} (S_0)^2 + \beta \frac{(\eta_1)^2}{2mc^2} (\vec{S})^2. \end{aligned} \quad (14)$$

We remark that this last equation is completely free of breaking parity terms. Nevertheless, a comparison between the equations (5) and (14) shows that the Hamiltonian described by (14) presents a torsion vector contribution in the kinetic part.

## 4 Equations of motion

We perform in this section the calculations of equations of motion. Let us begin by taking into account the two components spinor, described by (6). As explained in section (2), the next step is to write the Dirac equation in the Schrödinger form  $i\hbar \partial_t \psi = \mathcal{H} \psi$ . Straightforward calculations enable one to write the Hamiltonian for  $\varphi$  as

$$\mathcal{H}_\varphi^{tr} = \frac{1}{2m} (\vec{\Pi})^2 + B_0 + \vec{\sigma} \cdot \vec{Q}, \quad (15)$$

where

$$\begin{aligned} \vec{\Pi} &= \vec{p} - \frac{e}{c} \vec{A} - \frac{\eta_1}{c} S_0 \vec{\sigma} - \frac{\eta_1}{c} \sigma_5 \vec{S}, \quad B_0 = -\frac{(\eta_1)^2}{mc^2} (S_0)^2 + \frac{(\eta_1)^2}{2mc^2} (\vec{S})^2, \\ \vec{Q} &= \eta_1 \vec{S} - \frac{\hbar e}{2mc} \vec{B}, \end{aligned} \quad (16)$$

where  $\sigma_5 = (1/6) \varepsilon^{ijk} \sigma_i \sigma_j \sigma_k$ , see [24]. The expressions above are not exactly the same as derived in [7] and in [26] through the usual perturbative FWT. The basic difference is the term  $\sigma_5 \vec{S}$ . The appearance of this new term is based on advantage of using EFWT instead of FWT<sup>7</sup>.

It is important to note that the presence of terms of the kind  $\vec{S} \cdot \vec{B}$  in the transformed Hamiltonian (15) is related to the possibility of considering experimental tests of torsion field using magnetic resonance, as it was explained in [26]. Nevertheless, a straightforward comparison between equations (7) and (15) shows two differences between the SEFWT approach and the method presented here. The first one represents a new contribution in the kinetic part of (15) represented by a term<sup>8</sup> of the kind  $\sigma_5 \vec{S}$ . The second one is the absence, in the Hamiltonian (15), of a breaking parity term.

In order to quantize the Hamiltonian (16) and to write semi-classical equations of motion (After the calculus we make  $\hbar \rightarrow 0$  in the same procedure adopted in [27]). Let us consider the following relations

$$i\hbar \frac{d\hat{x}_i}{dt} = [\hat{x}_i, \mathcal{H}], \quad i\hbar \frac{d\hat{p}_i}{dt} = [\hat{p}_i, \mathcal{H}] \quad \text{and} \quad i\hbar \frac{d\hat{\sigma}_i}{dt} = [\hat{\sigma}_i, \mathcal{H}]. \quad (17)$$

So we get

$$\frac{d\hat{x}_i}{dt} = \frac{1}{m} \left( p_i - \frac{e}{c} A_i - \frac{\eta_1}{c} \sigma_i S_0 - \frac{\eta_1}{c} \sigma_5 S_i \right) = v_i \quad (18)$$

$$\frac{d\hat{p}_i}{dt} = \frac{\pi^j}{mc} \left( e \frac{\partial A_j}{\partial x^i} + \eta_1 S_0 \frac{\partial \sigma_j}{\partial x^i} + \eta_1 S_j \frac{\partial \sigma_5}{\partial x^i} \right) \quad (19)$$

$$\frac{d\hat{\sigma}_i}{dt} = \left[ \vec{R} \times \vec{\sigma} \right]_i, \quad (20)$$

<sup>7</sup>Using the Exact transformation, the risk of missing some important terms is lower.

<sup>8</sup>Observe that such term is a new one with relation to FWT [7] and SEFWT [20].

where

$$R_j = 2\frac{\eta_1}{\hbar}\left[S_j - \frac{1}{c}v_j S_0\right] - \frac{e}{mc}B_j, \quad (21)$$

and  $\sigma_5$  is the  $\gamma_5$  representation for the bi-spinor.

$$m\frac{dv_i}{dt} = \left[\vec{v} \times \vec{C}\right]_i + \frac{d}{dt}(u_i), \quad (22)$$

where

$$C_k = -\frac{e}{c}B_k - \frac{\eta_1}{c}\varepsilon_{klm}\frac{\partial}{\partial x_l}(S_0\sigma^m + \sigma_5 S^m) \quad \text{and} \quad u_i = -\frac{e}{c}A_i - \frac{\eta_1}{c}(S_0\sigma_i - \sigma_5 S_i). \quad (23)$$

The equation presented above represents the corrections for the classical Lorentz force acting on the Dirac particle. If one considers a trajectory described by this fermion, it is possible to observe that the terms with  $S_\mu$  could offer corrections for the path of the particle. These results are in accordance with the known equations of motion presented on [7].

Comparing the results for the equations of motion, that means, in this case to compare the exact approach with the semi-exact one, it is possible to see some differences. Looking one by one, we can note that the terms with  $S_i$  have different algebraic construction in (8) and (18). But in both equations they have the same physical meaning since it is mixed with the spinor matrices in first order (that is what matters for our phenomenological approach). Analogous considerations can be performed for equations (9) and (19), in which the unique difference is in the terms with  $\sigma_i$  and  $S_i$ . Finally, the equations (10) and (20) have no difference at all if we look for them carefully. The term of second order in torsion in (10) was considered neglectable in (20). The term with the vector product between  $S_i$  and  $v_i$  are zero because if we simply substitute  $v_i$  from (18) into this term, we can see that the term with  $A_i$  has the factor  $v/c^2$  (we are dealing with the nonrelativistic limit of the theory) and the others contribute only for the second order in torsion field. The term with the pure spatial part of momentum  $p_i$  will produce no physical difference when multiplied for the terms with torsion since each of these terms have a derivative of spin matrices with respect to the coordinates (It does not contribute for the trajectory of the particle, as it can be seen on equations (11) and (22)).

Another point that must be empathized is the necessity to extract from the exact transformed Hamiltonian the bound state of the theory, in order to propose possible experimental tests. As we know, the bound state would give us the possibility to use the powerful method presented in the series of papers [28, 29, 30, 31, 32] to find another possible experimental text for the torsion field using this theory. In the next section we present some comments and calculations about this relevant subject.

## 5 Bound state considerations

In this section we present brief considerations about the bound state of the Dirac Field interacting with space-time torsion. The perspective of CPT-Lorentz violation tests has considerable advantages in the context of Quantum Electrodynamics systems. There is, in fact, a set of examples related to atomic physics experiments (See [33, 34], and references cited therein). In

this sense, it is completely relevant the calculation of the bound state of the theory [32] we are considering here.

Let us consider the Lorentz violating potential  $V$  given by the following relation [34]

$$V = -\tilde{b}_j \sigma^j, \quad (24)$$

where  $\sigma$  is the spin matrices. The Lorentz potential comes from the equation (15) and considering such equation one can write the following bound state

$$\tilde{b}_j = b_j - \eta_1 S_j + \frac{\hbar e}{2mc} B_j. \quad (25)$$

In the last equation we observe the torsion contribution to the bound state. Such contribution is completely new and was not contemplated in the bound state associated to the EFWT for a Dirac theory related to the 80 CPT-Lorentz violating terms<sup>9</sup> [19]. However, although the possibility of indications of possible atomic experiments [31, 32] is related to the bound state (25), the magnitude of torsion field is irrelevant when compared, for example, with the magnitude of the magnetic field. For this reason, a concise proposal about experimental measurements of torsion field is not straightforward.

## 6 Discussion and conclusion

The nonrelativistic limit has been already studied for the Dirac field interacting with a set of external fields (except for the torsion field) in the context of CPT-Lorentz violation [19]. However, torsion case does not admit the usual exact transformation and the semi-exact transformation also seems to fail in such case. In this paper, we have derived a special technique that enables one to perform EFWT for the Dirac spinor field in the combined background of torsion and constant uniform magnetic fields. The Hamiltonian corresponding to the nonrelativistic limit was presented (15) and as one can check, it is completely free of braking parity terms.

We also have derived the equations of motion (18), (19) and (20) for the situation described above and the Lorentz force corrected by the presence of torsion field (22) was presented, together the discussion of possible experimental manifestations. We have calculated bound state of the Dirac field interacting with space-time torsion represented by the equation (25). However, due to the weakness of the torsion field, there is no a final conclusion that point out to the perspective about measuring the torsion field, using this technique. Notwithstanding, the main result here is the method itself, since it can be straightforwardly generalized in order to perform the EFWT for several cases until now not contemplated in the literature.

It is remarkable to say that the method presented in this work gives the possibility of performing the exact transformation for external fields not contemplated in [19]. In general, for each external field in the Hamiltonian (when performing EFWT is not possible) of the theory, there should be a particular special involution operator of the kind described in the equation (12). In this work we have considered torsion field. However, the search for such operators is a hard task and the study of a more general involution operator that contemplates all the possible external fields mentioned above should be in development, in a near future.

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<sup>9</sup>The reason is that, in [19] the criteria to perform EFWT is anti-commutation relation between the Hamiltonian and  $i\gamma^5\beta$ .



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